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CP-violation in $B \to \pi^+\pi^-$ and the unitarity triangle

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Abstract. We analyze the extraction of weak phases from CP-violation in $B \to \pi^+\pi^-$ decays. We propose to determine the unitarity triangle $(\bar{\rho}, \bar{\eta})$ by combining the information on mixing-induced CP-violation in $B \to \pi^+\pi^-$, S, with the precision observable $\sin 2\beta$ obtained from the CP-asymmetry in $B \to \psi K_S$. It is then possible to write down exact analytical expressions for $\bar{\rho}$ and $\bar{\eta}$ as simple functions of the observables S and $\sin 2\beta$ and of the penguin parameters r and ϕ. As an application clean lower bounds on $\bar{\eta}$ and $1-\bar{\rho}$ can be derived as functions of S and $\sin 2\beta$, essentially without hadronic uncertainty. Computing r and ϕ within QCD factorization yields precise determinations of $\bar{\rho}$ and $\bar{\eta}$ since the dependence on r and ϕ is rather weak. It is emphasized that the sensitivity to the phase ϕ enters only at second order and is extremely small for moderate values of this phase, predicted in the heavy-quark limit. Transparent analytical formulas are further given and discussed for the parameter C of direct CP-violation in $B \to \pi^+\pi^-$. Predictions and uncertainties for r and ϕ in QCD factorization are examined in detail. It is pointed out that a simultaneous expansion in $1/m_b$ and $1/N$ leads to interesting simplifications. At first order infrared divergences are absent, while the most important effects are retained. Independent experimental tests of the factorization framework are briefly discussed.

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1 Introduction

The main goal of the current experimental program at the SLAC and KEK B-meson factories is a stringent test of the standard model description of CP-violation. In the future this aim will be pursued with measurements of still higher precision from hadron machines at Fermilab and CERN. A crucial benchmark is the time-dependent CP-violation in $B \to \psi K_S$ decays, which allows us to infer the CKM phase β with negligible hadronic uncertainties. Likewise of central importance for obtaining additional information on the CKM parameters is the time-dependent CP-violation, both mixing-induced (S) and direct (C), in $B \to \pi^+\pi^-$. However, in this case the extraction of weak phases is complicated by a penguin component in the decay amplitude, which carries a weak phase different from the leading tree-level contribution. This leads to a dependence of the CP-asymmetries in $B \to \pi^+\pi^-$ on hadronic physics and to corresponding theoretical uncertainties. A possible strategy to circumvent this problem is the isospin analysis by Gronau and London [1], where also the branching ratios of $B^+ \to \pi^+\pi^0$, $B \to \pi^0\pi^0$ and their charge conjugates have to be measured. This method is theoretically very clean, but the difficulty to measure $B \to \pi^0\pi^0$ decays with sufficient accuracy and the existence of discrete ambiguities are likely to prevent a successful realization.

It is the purpose of this paper to demonstrate how the information on weak phases contained in the CP-asymmetries of $B \to \pi^+\pi^-$ itself can be extracted in an optimal way. To some extent theoretical input on the penguin-to-tree ratio will be needed and can be provided by the QCD factorization approach [2–4]. However, we will show that the impact of uncertainties in the calculation is in fact very mild. Moreover, even if the detailed predictions of QCD factorization are ignored, it is still possible to derive bounds on the CKM unitarity triangle, using only very conservative assumptions.

In order to derive these results we propose the following strategy. First, the time-dependent CP-asymmetries in $B \to \pi^+\pi^-$ are expressed in terms of the Wolfenstein parameters $\bar{\rho}$ and $\bar{\eta}$. At the same time purely hadronic quantities are systematically isolated from the CKM parameters, introducing the magnitude r and phase ϕ of a suitably normalized penguin-to-tree ratio [4]. We then combine the observable $S(\bar{\rho}, \bar{\eta}, r, \phi)$ with the accurately known value of $\sin 2\beta(\bar{\rho}, \bar{\eta})$ from $B \to \psi K_S$. This allows us to obtain the exact unitarity triangle, $\bar{\rho}$ and $\bar{\eta}$, in a simple analytical form, depending only on $\sin 2\beta$, S and the hadronic quantities r and ϕ. The dependence on the latter turns out to be particularly transparent, which greatly facilitates any further analysis. We are then able to derive bounds on the

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unitarity triangle practically free of hadronic uncertainties, or to fix $\bar\rho$ and $\bar\eta$ with theoretical input for r and ϕ.

There is already an extensive literature on the subject of extracting information on weak mixing angles from CP-violation in $B \to \pi^+\pi^-$ [5–15]. In these papers important aspects of the problem have been discussed and suggestions were made to constrain the theoretical uncertainties. Here we present a new way of exploiting the information contained in the CP-violation observables S and $\sin 2\beta$. The crucial elements are a definition of hadronic quantities r and ϕ independent of the CKM parameters, the direct formulation of the weak phases in terms of the basic Wolfenstein parameters $\bar\rho$, $\bar\eta$, the resulting analytical determination of the unitarity triangle and the exact, explicit and very simple dependence on r and ϕ. This in turn greatly facilitates the analysis of theoretical uncertainties and gives, in combination with the results based on the heavy-quark limit, robust determinations of the unitarity triangle, or CKM bounds with minimal hadronic input. These ideas were first presented in [16]. Subsequently, this analysis has been further discussed by Botella and Silva [17] and Lavoura [18].

This paper is organized as follows. In Sect. 2 we collect important basic formulas describing CP-violation in $B \to \pi^+\pi^-$. In Sect. 3 we discuss the theory of the penguin parameters r and ϕ in the framework of QCD factorization. Based on previous work we address in particular the issue of the theoretical uncertainties. In addition we investigate the analysis of $B \to \pi^+\pi^-$ decay amplitudes in a simultaneous expansion in both $1/m_b$ and $1/N$, where N is the number of colors. An interesting pattern of systematic simplifications resulting from the double expansion is pointed out. After this discussion of the hadronic input, we turn to our phenomenological analysis. Section 4 explores the determination of the unitarity triangle from S and $\sin 2\beta$ within the standard model. Simple analytical expressions are presented and theoretically clean bounds are derived. We also evaluate the standard model expectation for S using results from QCD factorization. Section 5 examines what can be learned from C, the parameter of direct CP-violation in $B \to \pi^+\pi^-$. Methods to validate the predictions of QCD factorization for $B \to \pi^+\pi^-$ using additional observables are reviewed in Sect. 6. We summarize our main results in Sect. 7.

2 Basic formulas

The time-dependent CP-asymmetry in $B \to \pi^+\pi^-$ decays is defined by

$$
\begin{aligned}
A^{\pi\pi}_{CP}(t) &= \frac{B(B(t)\to\pi^+\pi^-) - B(\bar B(t)\to\pi^+\pi^-)}{B(B(t)\to\pi^+\pi^-) + B(\bar B(t)\to\pi^+\pi^-)} \\
&= -S\,\sin(\Delta m_B t) + C\,\cos(\Delta m_B t), \qquad (1)
\end{aligned}
$$

where

$$
\begin{aligned}
S &= \frac{2\,\mathrm{Im}\,\xi}{1+|\xi|^2}, \\
C &= \frac{1-|\xi|^2}{1+|\xi|^2}, \qquad (2) \\
\xi &= \mathrm{e}^{-2\mathrm{i}\beta}\,\frac{\mathrm{e}^{-\mathrm{i}\gamma}+P/T}{\mathrm{e}^{+\mathrm{i}\gamma}+P/T}.
\end{aligned}
$$

In terms of the Wolfenstein parameters $\bar\rho$ and $\bar\eta$ [19,20] the CKM phase factors read

$$
\begin{aligned}
\mathrm{e}^{\pm\mathrm{i}\gamma} &= \frac{\bar\rho \pm \mathrm{i}\bar\eta}{\sqrt{\bar\rho^2+\bar\eta^2}}, \\
\mathrm{e}^{-2\mathrm{i}\beta} &= \frac{(1-\bar\rho)^2-\bar\eta^2-2\mathrm{i}\bar\eta(1-\bar\rho)}{(1-\bar\rho)^2+\bar\eta^2}. \qquad (3)
\end{aligned}
$$

The penguin-to-tree ratio P/T can be written as

$$
\frac{P}{T} = \frac{r\mathrm{e}^{\mathrm{i}\phi}}{\sqrt{\bar\rho^2+\bar\eta^2}}. \qquad (4)
$$

The real parameters r and ϕ defined in this way are pure strong interaction quantities without further dependence on the CKM variables.

For any given values of r and ϕ a measurement of S defines a curve in the ($\bar\rho$, $\bar\eta$) plane. Using the relations above, this constraint is given by the equation

$$
\begin{aligned}
S &= 2\bar\eta \qquad (5)\\
&\times \frac{[\bar\rho^2+\bar\eta^2-r^2-\bar\rho(1-r^2)+(\bar\rho^2+\bar\eta^2-1)r\cos\phi]}{((1-\bar\rho)^2+\bar\eta^2)(\bar\rho^2+\bar\eta^2+r^2+2r\bar\rho\cos\phi)}.
\end{aligned}
$$

Similarly the relation between C and $\bar\rho$, $\bar\eta$ reads

$$
C = \frac{2r\bar\eta\,\sin\phi}{\bar\rho^2+\bar\eta^2+r^2+2r\bar\rho\cos\phi}. \qquad (6)
$$

This is equivalent to

$$
\begin{aligned}
&(\bar\rho + r\cos\phi)^2 + \left(\bar\eta - \frac{r\sin\phi}{C}\right)^2 \\
&= \left(\frac{1}{C^2}-1\right)(r\sin\phi)^2, \qquad (7)
\end{aligned}
$$

describing a circle in the ($\bar\rho$, $\bar\eta$) plane with center at $(-r\cos\phi,\ (r\sin\phi)/C)$ and radius $r\sin\phi\sqrt{1-C^2}/C$.

The current experimental results for S and C are

$$
\begin{aligned}
S &= -0.30\pm0.17\pm0.03 \quad \text{(BaBar [21])}, \\
& -0.67\pm0.16\pm0.06 \quad \text{(Belle [22])}, \\
C &= -0.09\pm0.15\pm0.04 \quad \text{(BaBar [21])}, \\
& -0.56\pm0.12\pm0.06 \quad \text{(Belle [22])}. \qquad (8)
\end{aligned}
$$

Table 1. Theoretical values for r and ϕ and their uncertainties from various sources within QCD factorization. The upper part displays uncertainties from input into the factorization formulas. The lower part gives the uncertainties from a model estimate of power corrections (see text for details)

	μ	$m_u + m_d$	m_c	f_B	$F_0^{B\to\pi}$	α_2^π	λ_B	(ρ_H, ϕ_H)	(ρ_A, ϕ_A)
$r = 0.107$	± 0.005	± 0.019	± 0.002	± 0.003	± 0.002	± 0.003	± 0.002	± 0.001	± 0.024
$\phi = 0.150$	± 0.023	± 0.001	± 0.057	± 0.002	± 0.002	± 0.003	± 0.002	± 0.010	± 0.24

3 Penguin contribution

In this section we discuss the theoretical calculations of the penguin contribution in $B \to \pi^+\pi^-$. The analysis is based on the effective weak hamiltonian

$$\mathcal{H}_{\text{eff}} = \frac{G_{\text{F}}}{\sqrt{2}} \sum_{p=u,c} \lambda_p \left(C_1\, Q_1^p + C_2\, Q_2^p + \sum_{i=3,\ldots,6,\,8g} C_i\, Q_i \right) + \text{h.c.}, \tag{9}$$

where the C_i are Wilson coefficients, known at next-to-leading order [23], and Q_i $(i = 1, \ldots, 6)$ are local four-quark operators with flavor structure $(\bar{d}b)(\bar{q}q)$, $q = u, d, s, c, b$. Q_{8g} is the chromomagnetic operator $\sim m_b \bar{d}\sigma \cdot G(1+\gamma_5)b$. The CKM factors are here denoted by $\lambda_p = V_{pd}^* V_{pb}$.

3.1 QCD factorization

The penguin parameter $r\,\mathrm{e}^{\mathrm{i}\phi}$ has been computed in [4] in the framework of QCD factorization. The result can be expressed in the form

$$r\,\mathrm{e}^{\mathrm{i}\phi} = -\frac{a_4^c + r_\chi^\pi a_6^c + r_A[b_3 + 2b_4]}{a_1 + a_4^u + r_\chi^\pi a_6^u + r_A[b_1 + b_3 + 2b_4]}, \tag{10}$$

where we neglected the very small effects from electroweak penguin operators. The factorization coefficients a_i are linear combinations of the Wilson coefficients C_i in the effective weak Hamiltonian and include the $\mathcal{O}(\alpha_\text{s})$ corrections from hard gluon interactions in the weak matrix elements. Their expressions can be found in [4]. The quantities r_χ^π and r_A are defined by

$$\begin{aligned} r_\chi^\pi(\mu) &= \frac{2m_\pi^2}{\bar{m}_b(\mu)(\bar{m}_u(\mu) + \bar{m}_d(\mu))}, \\ r_A &= \frac{f_B f_\pi}{m_B^2 F_0^{B\to\pi}(0)}. \end{aligned} \tag{11}$$

r_χ^π is defined in terms of the $\overline{\text{MS}}$ quark masses $\bar{m}_q(\mu)$ and depends on the renormalization scale μ. $F_0^{B\to\pi}(0)$ is a $B \to \pi$ transition form factor, evaluated at momentum transfer $q^2 = m_\pi^2 \simeq 0$.

Both quantities in (11) are of subleading power, $r_\chi^\pi \sim r_A \sim \Lambda_{\text{QCD}}/m_b$. $r_A \approx 0.003$ is numerically very small. It sets the scale for the weak annihilation effects in the amplitude, which are parametrized by the b_i in (10) [4]. They represent power corrections that are not calculable in QCD factorization. Model-dependent estimates for these subleading effects have been given in [4] in order to assess the corresponding uncertainties. On the other hand, $r_\chi^\pi(1.5\,\text{GeV}) \approx 0.7$ is numerically sizable. Still the important penguin contributions a_6^p, $p = u, c$, are calculable and can be included in the analysis. A third class of power corrections that need to be considered are uncalculable spectator interactions, some of which also come with the parameter r_χ^π. These enter the a_i in (10) and were also estimated in [4].

In Table 1 we show the values for r and ϕ from a calculation within the QCD factorization framework as described in [4]. We also display the uncertainties from various sources, distinguishing two classes. In the upper part we give the uncertainties from input into the factorization formulas at next-to-leading order, as well as the sensitivity to the renormalization scale μ. This input is defined in Table 2. The second class of uncertainty is due to the model estimates employed for power corrections. As in [4] these effects are parameterized by phenomenological quantities,

$$X_{H,A} = \left(1 + \rho_{H,A}\,\mathrm{e}^{\mathrm{i}\phi_{H,A}}\right) \ln\frac{m_B}{\Lambda_h}, \tag{12}$$

that enter power corrections to hard spectator scattering (H) and weak annihilation effects (A). The default values have $\rho_{H,A} = 0$. They depend on an infrared cut-off parameter Λ_h, which we take as $\Lambda_h = 0.5\,\text{GeV}$. An error of 100% is then assigned to this estimate by allowing for arbitrary phases ϕ_H, ϕ_A and taking ρ_H, ρ_A between 0 and 1. The impact on r and ϕ of this second class of uncertainties is shown in the lower part of Table 1 and is seen to be completely dominated by the annihilation contributions.

Adding the errors in quadrature we find

$$r = 0.107 \pm 0.020 \pm 0.024, \tag{13}$$

$$\phi = 0.15 \pm 0.06 \pm 0.24, \tag{14}$$

where the first (second) errors are from the first (second) class of uncertainties. Combining both in quadrature we

Table 2. Input used for Table 1. We take $\mu \in [m_b/2, 2m_b]$. The values for $m_u + m_d \equiv (m_u + m_d)(2\,\text{GeV})$, $m_c(m_b)$, f_B and λ_B are in GeV

$m_u + m_d$	$m_c(m_b)$	f_B	$F_0^{B\to\pi}(0)$	α_2^π	λ_B
0.0091 ± 0.0021	1.3 ± 0.2	0.18 ± 0.04	0.28 ± 0.05	0.1 ± 0.3	0.35 ± 0.15

finally arrive at

$$r = 0.107 \pm 0.031, \quad \phi = 0.15 \pm 0.25, \tag{15}$$

which we take as our reference predictions for r and ϕ in QCD factorization.

3.2 Expansion in $1/m_b$ and $1/N$

In order to obtain additional insight into the structure of hadronic B-decay amplitudes, it will be interesting to consider these quantities in a simultaneous expansion in powers of $1/m_b$ and $1/N$, where N is the number of colors. Expanding in $1/m_b$ alone corresponds to the framework of QCD factorization, implying naive factorization at leading order, which receives calculable corrections. A large-N expansion of the weak decay amplitudes gives an entirely different justification for naive factorization, which holds at leading order in $1/N$. The large-N limit has previously been applied to weak decays of kaons and D mesons [24–26]. Here we would like to explore the consequences of combining the heavy-quark and large-N limits in the analysis of important subleading corrections to naive factorization.

For this purpose we treat the Wilson coefficients C_1, C_2, C_4, C_6 and C_{8g} as quantities of order one. In the limit $N \to \infty$ strictly speaking only C_1 and C_{8g} are non-vanishing. However, C_2, C_4 and C_6 vanish slower than $1/N$, as $N \to \infty$, if the large logarithm $\ln M_W/m_b$ is considered $\sim N$, in accordance with the usual renormalization group (RG) counting $\alpha_s \ln M_W/m_b \sim 1$ and with $\alpha_s \sim 1/N$. More specifically, C_2, $C_4 \sim \ln N/N$ and $C_6 \sim 1/N^{2/11}$. The formal treatment of these coefficients as order unity is identical to the usual counting of the coefficients in RG improved perturbation theory. From this latter case it is also clear that the small numerical size of C_2 and especially of the penguin coefficients C_4, C_6 is related to small anomalous dimensions, which are small accidentally, but not because of a particular parametric suppression. The coefficients C_3, C_5 on the other hand are suppressed relative to C_4 and C_6 by an explicit factor of N. We will thus take $C_3, C_5 \sim 1/N$. Similar considerations can be found in [25,26]. Concerning the heavy-quark limit $m_b \gg \Lambda_{\rm QCD}$ there is no difference to the conventional counting in RG improved perturbation theory, where $M_W \gg m_b$ is assumed.

After discussing the Wilson coefficients we next turn to the hadronic matrix elements in QCD factorization. We first need to determine how the various quantities entering these matrix elements scale for large m_b and N. Based on these results we shall then expand the factorization coefficients a_i and b_i to next-to-leading order in a double expansion in $1/m_b$ and $1/N$. That is, we keep terms of order one, as well as corrections suppressed by either a single power of $1/m_b$ or $1/N$. We neglect terms that exhibit a suppression by two or more powers of the expansion parameters $1/m_b$ or $1/N$, such as $1/m_b^2$, $1/N^2$ and $1/m_bN$. It turns out that in this approximation all subleading contributions suffering from infrared end-point singularities in the QCD factorization approach are absent. This includes both spectator interactions of subleading twist and all weak annihilation amplitudes. On the other hand, important effects as hard QCD corrections or the penguin contribution from Q_6, which is formally power suppressed, are retained.

Let us postpone annihilation effects for the moment and examine first the coefficients a_i. The most important ones for our purpose are a_1, a_4 and a_6, which may be written as [4]

$$a_1 = C_1 + \frac{C_2}{N}\left[1 + \frac{C_F\alpha_s}{4\pi}V_\pi\right] + \frac{C_2}{N}\frac{C_F\pi\alpha_s}{N}H_{\pi\pi}, \tag{16}$$

$$a_4^p = C_4 + \frac{C_3}{N}\left[1 + \frac{C_F\alpha_s}{4\pi}V_\pi\right] + \frac{C_F\alpha_s}{4\pi}\frac{P^p_{\pi,2}}{N} + \frac{C_3}{N}\frac{C_F\pi\alpha_s}{N}H_{\pi\pi}, \tag{17}$$

$$a_6^p = C_6 + \frac{C_5}{N}\left[1 - 6\frac{C_F\alpha_s}{4\pi}\right] + \frac{C_F\alpha_s}{4\pi}\frac{P^p_{\pi,3}}{N}. \tag{18}$$

Here V_π, $P^p_{\pi,2}$, $P^p_{\pi,3}$ are calculable quantities of order one (in both $1/m_b$ and $1/N$), containing convolution integrals over pion light-cone distribution amplitudes [4]. The coefficient $H_{\pi\pi}$ describes hard spectator scattering and reads

$$\begin{aligned} H_{\pi\pi} &= \frac{f_B f_\pi}{m_B^2 F_0^{B\to\pi}(0)}\int_0^1 \frac{d\xi}{\xi}\phi_B(\xi)\int_0^1\frac{dx}{\bar x}\phi_\pi(x) \\ &\quad\times\int_0^1\frac{dy}{\bar y}\left[\phi_\pi(y) + r_\chi^\pi\frac{\bar x}{x}\phi_p(y)\right] \\ &\equiv H_{\pi\pi,2} + H_{\pi\pi,3}. \end{aligned}\tag{19}$$

Here ϕ_B is the leading-twist light-cone distribution amplitude of the B meson, ϕ_π the one of the pion. $\phi_p(y) = 1$ is the two-particle, twist-3 component of the pion light-cone wave function. We recall that the correction $\sim r_\chi^\pi\phi_p$ (defined as $H_{\pi\pi,3}$ in (19)) is uncalculable, as indicated by the end-point divergence in the y-integral, but it is power suppressed in $1/m_b$.

The large-m_b, large-N scaling of the various terms is as follows:

$$\phi_B, \phi_\pi, \phi_p \sim 1, \quad \xi \sim 1/m_b, \quad x, y \sim 1. \tag{20}$$

$$\begin{aligned} &r_\chi^\pi \sim 1/m_b, \quad F_0^{B\to\pi}(0) \sim 1/m_b^{3/2}, \\ &f_B \sim N^{1/2}/m_b^{1/2}, \quad f_\pi \sim N^{1/2}. \end{aligned}\tag{21}$$

We then have

$$H_{\pi\pi,2} \sim N, \quad H_{\pi\pi,3} \sim N/m_b. \tag{22}$$

Further we note $C_F \sim N$, $\alpha_s \sim 1/N$. Expanding (16)–(18) to first order in $1/m_b$ and $1/N$ we find

$$a_1 \doteq C_1 + \frac{C_2}{N}\left[1 + \frac{C_F\alpha_s}{4\pi}V_\pi\right] + \frac{C_2}{N}\frac{C_F\pi\alpha_s}{N}H_{\pi\pi,2}, \tag{23}$$

$$a_4^p \doteq C_4 + \frac{C_F\alpha_s}{4\pi}\frac{P^p_{\pi,2}}{N}, \quad r_\chi^\pi a_6^p \doteq r_\chi^\pi C_6. \tag{24}$$

We observe that to this order in the double expansion the uncalculable power correction $\sim H_{\pi\pi,3}$ does not appear in

a_1, to which it only contributes at order $1/m_b N$. On the other hand, the leading-twist effect $\sim H_{\pi\pi,2}$ is retained as well as the vertex corrections $\sim V_{\pi\pi}$. Both contribute to a_1 at order $1/N$. For a_4^p the hard spectator term is absent altogether because it scales as $1/N^2$, but the non-trivial penguin loop corrections still contribute at order $1/N$. Since $r_\chi^\pi a_6^p$ is already $\sim 1/m_b$ we omit all $1/N$ effects.

We next show that within our approximation the expressions in (23) and (24) receive no further corrections from weak annihilation. From [4] we recall that the annihilation coefficients b_i appearing in (10) can be written as

$$b_1 = \frac{C_F}{N^2} C_1 A_1^i, \quad b_4 = \frac{C_F}{N^2}\left[C_4 A_1^i + C_6 A_2^i\right], \tag{25}$$

$$b_3 = \frac{C_F}{N^2}\left[C_3 A_1^i + C_5(A_3^i + A_3^f) + N C_6 A_3^f\right]. \tag{26}$$

The parameters $A_k^{i,f}$ are not calculable in QCD factorization. However, they have been estimated in [4] from a diagrammatic analysis of the annihilation topologies in a way that keeps track of the correct counting in $1/m_b$ and $1/N$. One finds [4]

$$A_{1,2}^i \sim 1/N, \quad A_3^{i,f} \sim 1/m_b N, \quad r_A \sim N/m_b, \tag{27}$$

and therefore

$$r_A b_{1,4} \sim 1/m_b N, \quad r_A b_3 \sim 1/m_b^2. \tag{28}$$

All annihilation effects thus contribute to r in (10) only at second order in the double expansion, as anticipated above. Numerically the largest impact on r comes from b_3, which in turn is dominated by the term $\sim C_6$. The contribution to b_3 from C_3 is highly suppressed, $\sim 1/m_b N^2$, the one from C_5 even by $1/m_b^2 N^2$, and both are very small numerically. We remark that, unlike all other corrections to naive factorization, the annihilation term $r_A b_3$ is leading in the large-N limit.

We conclude that (23) and (24) give indeed the amplitude coefficients complete through first order in $1/m_b$ and $1/N$. We stress again that in this approximation these quantities are fully calculable. In other words, the problematic corrections, uncalculable in QCD factorization, from higher-twist spectator interactions and weak annihilation are at least doubly suppressed in the combined heavy-quark, large-N expansion. This observation shows that the large-N limit yields a useful organizing principle complementary to the $1/m_b$ expansion. If the large-N limit is not entirely unrealistic, the double expansion will provide an additional tool to improve our theoretical control over two-body hadronic B-decay amplitudes. Experience with similar decays of K and D mesons suggests that considerations based on large-N arguments can be a reasonable approach to these problems [24–26]. After all, as discussed in this paper, for some applications approximate results are sufficient and precise calculations are not necessarily required.

It is interesting to evaluate the approximations (23) and (24) numerically and to compare with the full NLO QCD factorization results that include the estimates of uncalculable power corrections. Using default input parameters we quote three values for the various coefficients: The first, second and third numbers give, respectively, the result for QCD factorization, for the $1/m_b$, $1/N$ approximation in (23) and (24), and for naive factorization.[1] We have

$$\begin{aligned}
a_1 &= 1.00 + 0.02\mathrm{i};\ 1.02 + 0.02\mathrm{i};\ 1.03;\\
a_4^c &= -0.033 - 0.007\mathrm{i};\ -0.038 - 0.006\mathrm{i};\ -0.027;\\
r_\chi^\pi a_6^c &= -0.056 - 0.007\mathrm{i};\ -0.041;\ -0.038;\\
r &= 0.107;\ 0.084;\ 0.068;\\
\phi &= 0.150;\ 0.065;\ 0.
\end{aligned} \tag{29}$$

For comparison, the default value for the annihilation correction to the penguin amplitude is $r_A(b_3 + 2b_4) = -0.010$ in the model of [4].

We see that to first order in the double expansion the values are qualitatively similar to the QCD factorization result. Since we use the full NLO coefficient C_4 in a_4^c, there is strictly speaking an ambiguity whether or not to include in a_4^c the C_3 terms, which cancel a small part of the NLO scheme dependence. If we include C_3 in the vertex and penguin correction part of a_4^c, still neglecting hard spectator scattering, we find $-0.036 - 0.007\mathrm{i}$ instead of the $-0.038 - 0.006\mathrm{i}$ above, and (r, ϕ) becomes $(0.081, 0.083)$ instead of $(0.084, 0.065)$. Apparently the $1/N$ approximation proposed here does not seem to be entirely unrealistic, at least within the standard QCD factorization framework at next-to-leading order.

4 Unitarity triangle from S and $\sin 2\beta$

4.1 Determining $\bar\rho$ and $\bar\eta$

In this section we discuss the determination of the unitarity triangle by combining the information from S with the value of $\sin 2\beta$, which is known with high precision from CP-violation measurements in $B \to J/\Psi K_S$. As we shall see, this method allows for a particularly transparent analysis of the various uncertainties. Both $\bar\rho$ and $\bar\eta$ can be obtained, which fixes the unitarity triangle. A comparison with other determinations then provides us with a test of the standard model.

The angle β of the unitarity triangle is given by

$$\tau \equiv \cot\beta = \frac{\sin 2\beta}{1 - \sqrt{1 - \sin^2 2\beta}}. \tag{30}$$

We shall take the value [27]

$$\sin 2\beta = 0.739 \pm 0.048, \tag{31}$$

which implies

$$\tau = 2.26 \pm 0.22. \tag{32}$$

Given a value of τ, $\bar\rho$ is related to $\bar\eta$ by

$$\bar\rho = 1 - \tau\,\bar\eta. \tag{33}$$

[1] The coefficients in naive factorization are $a_i = C_i + C_{i-(-1)^i}/3$, with leading-log values for the C_i, and $b_i = 0$.

The parameter $\bar\rho$ may thus be eliminated from S in (5), which can be solved for $\bar\eta$ to yield

$$\bar\eta = ((1+\tau S)(1+r\cos\phi) - \left((1-S^2)(1+r^2+2r\cos\phi) - (1+\tau S)^2 r^2 \sin^2\phi\right)^{1/2}\Big) / \left((1+\tau^2)S\right). \quad (34)$$

So far, no approximations have been made and (33) and (34) are still completely general. The two observables τ (or $\sin 2\beta$) and S determine $\bar\eta$ and $\bar\rho$ once the theoretical penguin parameters r and ϕ are provided. It is at this point that some theoretical input is necessary. We will now consider the impact of the parameters r and ϕ, and of their uncertainties, on the analysis.

We first would like to point out that the sensitivity of $\bar\eta$ in (34) to the strong phase ϕ is rather mild. In fact, the dependence on ϕ enters in (34) only at second order. Expanding in ϕ we obtain to lowest order

$$\bar\eta \doteq \frac{1+\tau S-\sqrt{1-S^2}}{(1+\tau^2)S}(1+r). \quad (35)$$

This result is corrected at second order in ϕ through

$$\Delta\bar\eta = \left(\frac{1-S^2+r(1+\tau S)^2}{(1+r)\sqrt{1-S^2}} - (1+\tau\ S)\right)\frac{r\phi^2}{2(1+\tau^2)S}. \quad (36)$$

This feature is very welcome since it is in particular the strong phase that is difficult to calculate with good precision. Nevertheless, we know from factorization in the heavy-quark limit that the strong phase is suppressed either by α_s, if it arises from hard scattering, or by Λ_{QCD}/m_b for soft corrections. This means that even if ϕ is not accurately known, it will have very little impact on $\bar\eta$ as long as it is of moderate size. Since r is also small, the second order effect from ϕ in (36) is even further reduced. As an example, for $S=0$ one finds

$$\Delta\bar\eta = -\frac{1-r}{1+r}\,\frac{\tau}{1+\tau^2}\,\frac{r}{2}\phi^2. \quad (37)$$

For typical values $r\approx 0.1$, this implies that $|\Delta\bar\eta| < 0.01$ for ϕ up to 45°, which is already a large phase. For $\phi < 20°$, which is more realistic, one has a negligible shift $|\Delta\bar\eta| < 0.002$. Consequently, the relation in (35) is most likely a very good approximation to the exact result. Note that apart from neglecting the phase ϕ, no approximation is made in (35). The resulting expression is strikingly simple. $\bar\eta$ is essentially determined by the CP-violating observables S and τ. The only dependence on the penguin parameter r is through an overall factor of $(1+r)$. Typically $r\approx 0.1$, as predicted in QCD factorization but indicated also by other approaches. The effect is again a fairly small correction. A 100% uncertainty on this estimate of r would translate into a 10% uncertainty in $\bar\eta$.

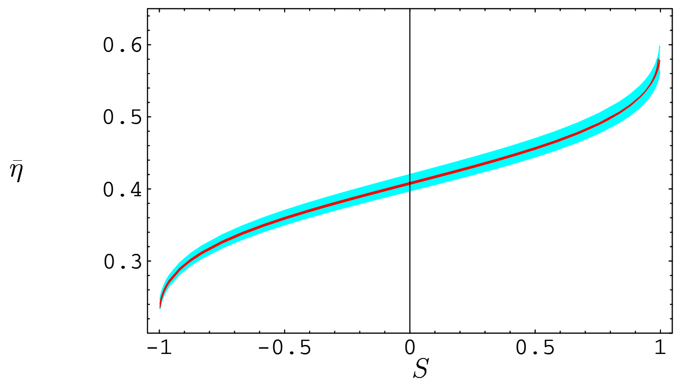

Fig. 1. CKM phase $\bar\eta$ as a function of the mixing-induced CP-asymmetry S in $B\to\pi^+\pi^-$ within the standard model for $\sin 2\beta = 0.739$. The dark (light) band reflects the theoretical uncertainty in the penguin phase $\phi = 0.15 \pm 0.25$ (penguin amplitude $r = 0.107 \pm 0.031$)

The determination of $\bar\eta$ as a function of S is shown in Fig. 1, which displays the theoretical uncertainty from the penguin parameters r and ϕ in QCD factorization.

In the determination of $\bar\eta$ and $\bar\rho$ described here discrete ambiguities do in principle arise. One source is the well-known ambiguity in relating $\sin 2\beta$ to a value of β, or equivalently $\tau = \cot\beta$. Apart from the solution shown in (30), a second solution exists with the sign of the square root reversed. It corresponds to a larger value of β, incompatible with the standard fit of the unitarity triangle. An additional ambiguity comes from the second solution for $\bar\eta$, which is the result given in (34) with a positive sign in front of the square root. This case may be considered separately, but will usually also yield solutions in conflict with other information on the CKM phases.

4.2 Standard model prediction for S

In this section we shall use theoretical information on r and ϕ based on QCD factorization to compute the value of S expected within the standard model. Using (33), one can write (5) in the form

$$S = 2\,\frac{\frac{1+\tau^2}{1+r\cos\phi}\,\bar\eta - \tau\left(1+\frac{r^2\sin^2\phi}{(1+r\cos\phi)^2}\right)}{1+\left(\frac{1+\tau^2}{1+r\cos\phi}\,\bar\eta-\tau\right)^2 + (1+\tau^2)\frac{r^2\sin^2\phi}{(1+r\cos\phi)^2}}. \quad (38)$$

Since the terms $\sim r^2\sin^2\phi$ $(< 10^{-3})$ are very small, S is well approximated by

$$S = \frac{2z}{1+z^2}, \quad z \equiv \frac{1+\tau^2}{1+r\cos\phi}\,\bar\eta - \tau. \quad (39)$$

Taking [28]

$$\tau = 2.26 \pm 0.22, \quad \bar\eta = 0.35 \pm 0.04, \quad (40)$$

and

$$r = 0.107 \pm 0.031, \quad \phi = 0.15 \pm 0.25, \quad (41)$$

we find from (38)

$$S = -0.59\,^{+0.18}_{-0.11}\,(\tau)\,^{+0.38}_{-0.25}\,(\bar\eta)\,^{-0.07}_{+0.08}\,(r)\,^{+0.02}_{-0.00}\,(\phi). \qquad (42)$$

We note that the uncertainty from the hadronic phase ϕ is negligible and the uncertainty from the penguin parameter r is rather moderate. The error from τ or $\sin 2\beta$, which reflects the current experimental accuracy in this quantity, is considerably larger. The dominant uncertainty, however, is due to $\bar\eta$, which for the purpose of predicting S has here been taken from a standard CKM fit. Clearly, the large sensitivity of S to $\bar\eta$ is equivalent to the fact that in turn $\bar\eta$ has only a fairly weak dependence on S. This feature was already discussed in [29]. Equation (42) shows that the standard model prefers negative values for S, but it is difficult to obtain an accurate prediction.

4.3 CKM bounds with minimal hadronic input

We will now relax the constraints on the penguin parameters r and ϕ coming from direct theoretical calculations and derive bounds on $\bar\rho$ and $\bar\eta$ without relying on any detailed information about hadronic quantities. Specifically, we shall only assume that the strong phase ϕ fulfills

$$-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}. \qquad (43)$$

In view of the fact that ϕ is systematically suppressed in the heavy-quark limit, and that typically $\phi \approx 0.2$ from QCD factorization, this assumption is very weak.

As has been shown in [16], the following inequality can be derived from (34) for $-\sin 2\beta \leq S \leq 1$:

$$\bar\eta \geq \frac{1+\tau S - \sqrt{1-S^2}}{(1+\tau^2)S}(1+r\cos\phi). \qquad (44)$$

This bound is still *exact* and requires no information on the phase ϕ. (The only condition is that $(1+r\cos\phi)$ is positive, which is no restriction in practice.)

Assuming now (43), we have $1+r\cos\phi \geq 1$ and

$$\bar\eta \geq \frac{1+\tau S - \sqrt{1-S^2}}{(1+\tau^2)S} \quad \text{if} \quad -\sin 2\beta \leq S \leq 1. \qquad (45)$$

We emphasize that this lower bound on $\bar\eta$ depends only on the observables τ and S and is essentially free of hadronic uncertainties. It holds in the standard model and it is effective under the condition that S will eventually be measured in the interval $[-\sin 2\beta, 1]$. Since both r and ϕ are expected to be quite small, we anticipate that the lower limit (45) is a fairly strong bound, close to the actual value of $\bar\eta$ itself (compare (35)). We also note that the lower bound (45) represents the solution for the unitarity triangle in the limit of vanishing penguin amplitude, $r=0$. In other words, the bounds for $\bar\eta$ and $\bar\rho$ are simply obtained by ignoring penguins and taking $S \equiv \sin 2\alpha$ when fixing the unitarity triangle from S and $\sin 2\beta$.

Let us briefly comment on the second solution for $\bar\eta$, which has the minus sign in front of the square root in

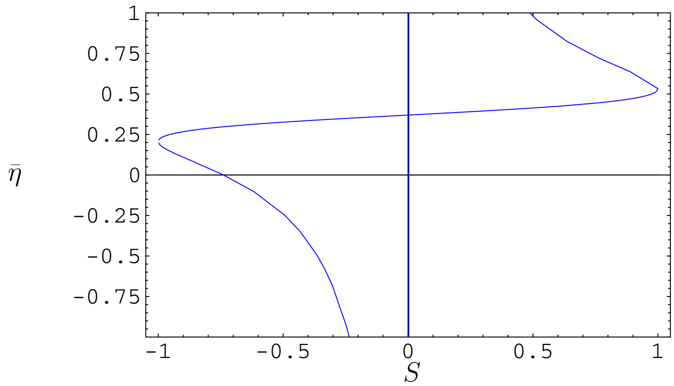

Fig. 2. Discrete ambiguities for $\bar\eta$ as a function of S with $\sin 2\beta = 0.739$, $r=0$. For $-\sin 2\beta \leq S \leq 1$ the middle branch defines the lower bound on $\bar\eta$, which is not affected by the additional solution

(34) replaced by a plus sign. For positive S this solution is always larger than (34) and the bound (45) is unaffected. For $-\sin 2\beta \leq S \leq 0$ the second solution gives a negative $\bar\eta$, which is excluded by independent information on the unitarity triangle (for instance from indirect CP-violation in neutral kaons (ε_K)). The additional solution for $\bar\eta$ is illustrated in Fig. 2 for $r=0$, the case relevant for the lower bound.

Because we have fixed the angle β, or τ, the lower bound on $\bar\eta$ is equivalent to an upper bound on $\bar\rho = 1 - \tau\bar\eta$. The constraint (45) may also be expressed as a lower bound on the angle γ:

$$\gamma \geq \frac{\pi}{2} - \arctan\frac{S - \tau(1-\sqrt{1-S^2})}{\tau S + 1 - \sqrt{1-S^2}}, \qquad (46)$$

or a lower bound on R_t:

$$R_t \equiv \sqrt{(1-\bar\rho)^2 + \bar\eta^2} \geq \frac{1+\tau S - \sqrt{1-S^2}}{\sqrt{1+\tau^2}S}. \qquad (47)$$

In Figs. 3 and 4 we represent the lower bound on $\bar\eta$ and γ as a function of S for various values of $\sin 2\beta$. From Fig. 3 we observe that the lower bound on $\bar\eta$ becomes stronger as either S or $\sin 2\beta$ increase. The sensitivity to $\sin 2\beta$ is less pronounced for the bound on γ. Similarly to $\bar\eta$, the

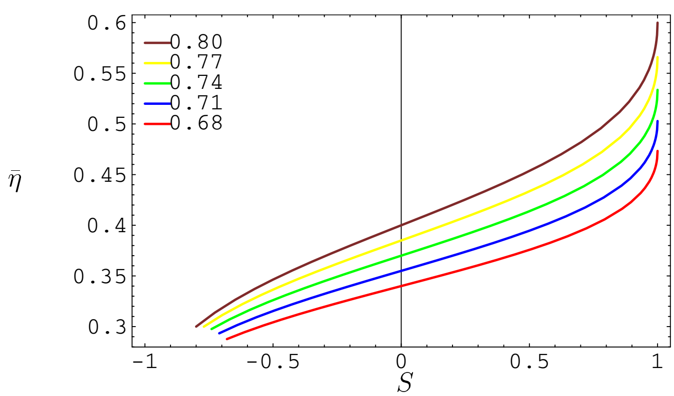

Fig. 3. Lower bound on $\bar\eta$ as a function of S for various values of $\sin 2\beta$ (increasing from bottom to top)

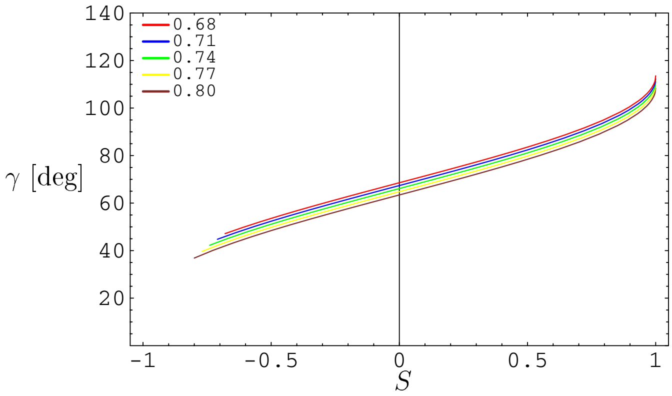

Fig. 4. Lower bound on γ as a function of S for various values of $\sin 2\beta$ (decreasing from bottom to top)

minimum allowed value for γ increases with S. A lower limit $\gamma = 90°$ is reached for $S = \sin 2\beta$.

In Fig. 5 we illustrate the region in the $(\bar\rho, \bar\eta)$ plane that can be constrained by the measurement of $\sin 2\beta$ and S using the bound in (45).

We finally note that the condition $r \cos\phi > 0$, which is crucial for the bound, could be independently checked [30] by measuring the mixing-induced CP-asymmetry in $B_s \to K^+K^-$. This is because the hadronic physics of $B_s \to K^+K^-$ is related to $B_d \to \pi^+\pi^-$ by U-spin symmetry, a feature that has already been employed for CKM phenomenology [31]. Our purpose here is to use information from $B_s \to K^+K^-$ in order to obtain additional input for $B_d \to \pi^+\pi^-$ within the approach suggested above. To this end we write the CP-violation observable S, defined in analogy to (1), for the case of $B_s \to K^+K^-$. Neglecting the small phase of B_s–$\bar B_s$ mixing, we find

$$S(B_s \to K^+K^-) = \frac{2\bar\eta\,(kr\cos\phi - \bar\rho)}{(kr\cos\phi - \bar\rho)^2 + \bar\eta^2 + k^2r^2\sin^2\phi}, \quad (48)$$

where

$$k \equiv \frac{1-\lambda^2}{\lambda^2} \approx 20 \quad (49)$$

for Wolfenstein parameter $\lambda = 0.22$. Because of the different CKM hierarchy of the $b \to s$ transition, the penguin contribution $\sim r$ is strongly enhanced by a factor of $k \approx 20$ compared to the case of $B_d \to \pi^+\pi^-$. On the other hand, the purely hadronic quantities r and ϕ are identical to the corresponding parameters for $B_d \to \pi^+\pi^-$ in the limit of exact U-spin symmetry. We notice that the presumably largest effects from U-spin breaking, coming from the difference of the decay constants f_K, f_π and the form factors for the $B_s \to K$ and $B_d \to \pi$ transitions, largely cancel in the ratio r of penguin-over-tree amplitudes. Explorations of further sources of U-spin breaking can be found in [30,32]. We shall assume that $r \approx 0.1$ as indicated by QCD factorization. Then the relevant penguin parameter in (48) is $kr \approx 2$, which dominates the much smaller values for $\bar\rho \sim 0.15$. As a consequence, (48) predicts, in the standard model, the sign of $S(B_s \to K^+K^-)$ in correspondence with the sign of $r\cos\phi$. In QCD factorization this sign is positive and one expects

$$S(B_s \to K^+K^-) \approx \bar\eta. \quad (50)$$

A future measurement of this observable will then provide a test of the assumption made to obtain the above bounds. We remark that also from other charmless hadronic B decays a sizable penguin amplitude is required, independently of detailed QCD calculations, where $r = 0.1$ is a typical value. It is thus basically excluded that kr will be much below about 2 and that the term $-\bar\rho$ in the numerator of (48) will be able to compete so as to change the conclusion. On the other hand, an extreme value of the phase $\phi \approx \pi/2$, for instance, could be indicated through the observation of a very small $S(B_s \to K^+K^-)$, which would typically amount to a few percent. (In the approximation above one would obtain $S(B_s \to K^+K^-) \approx -\bar\eta\bar\rho/2$, but the mixing phase could then no longer be neglected. It would tend to further reduce the asymmetry.)

In this comparison it is legitimate to assume the validity of the standard model throughout, as the strategy is to look for new physics via inconsistencies under this assumption, exploiting a multitude of experimental results and eliminating hadronic uncertainties.

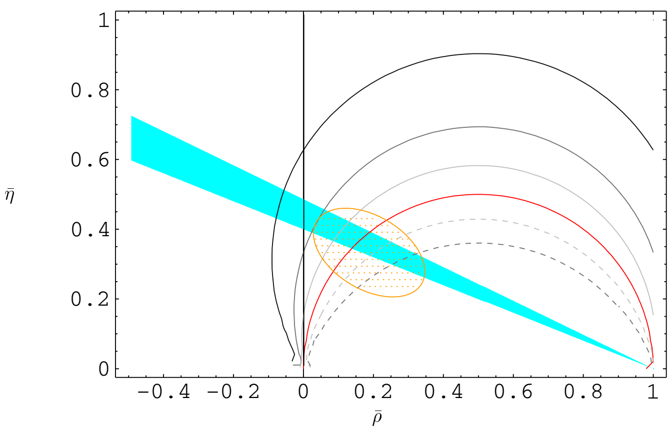

Fig. 5. Region in the $(\bar\rho, \bar\eta)$ plane constrained by $\sin 2\beta = 0.739 \pm 0.048$ (shaded sector) and various possible values for S. The allowed area is the part of the shaded sector to the left of a given line defined by S. These lines correspond, from bottom to top, to $S = -0.6$, -0.3, 0, 0.3, 0.6 and 0.9. The bound becomes stronger with increasing S. The result of a standard unitarity triangle fit (dotted ellipse, from [28]) is overlaid for comparison

5 Direct CP-violation

So far we have considered the implications of mixing-induced CP-violation, described by S. In the following we shall investigate how useful additional information can be extracted from a measurement of the direct CP-violation parameter C. An alternative discussion of this question can be found in [10].

The observable C (see (6)) is an odd function of ϕ. It is therefore sufficient to restrict the discussion to positive values of ϕ. A positive phase ϕ is obtained by the perturbative estimate in QCD factorization, neglecting soft phases with power suppression. For positive ϕ also C will be positive, assuming $\bar{\eta} > 0$, and a sign change in ϕ will simply flip the sign of C.

In contrast to the case of S, the hadronic quantities r and ϕ play a prominent role for C, as can be seen in (6). This will in general complicate the interpretation of an experimental result for C. One aspect of this can be seen as follows. It is usually expected that small values of the weak phase $\bar{\eta}$ and the strong phase ϕ correspond to a small CP-asymmetry C. However, in principle this need not be the case. As a counterexample let us consider the scenario where $\bar{\rho} = -0.35$, $\bar{\eta} = 0.07$, $r = 0.35$ and $\phi = 0.2 \approx 11°$. Of course this implies a very large angle γ, but this would be possible if the presence of new physics invalidates the standard unitarity triangle analysis (still the constraint from R_b is obeyed). Although both $\bar{\eta}$ and the strong phase ϕ are very small, these numbers give $C = 0.995$. More generally, such a situation occurs if $\bar{\rho} = -r$, leading to a cancellation in the denominator of C. In this case, assuming also that ϕ is small, we get

$$C \approx \frac{2\bar{\eta} r \phi}{\bar{\eta}^2 + (r\phi)^2}, \tag{51}$$

which takes its maximal value $C = 1$ for $\bar{\eta} = r\phi$. Clearly, a scenario of this type requires a very peculiar coincidence and may seem unlikely. Nevertheless the example illustrates that the proper interpretation of C can be rather involved.

The analysis of C becomes more transparent if we fix the weak parameters and study the impact of r and ϕ. An important application is a test of the standard model, obtained by taking $\bar{\rho}$ and $\bar{\eta}$ from a standard model fit and comparing the experimental result for C with the theoretical expression as a function of r and ϕ.

Let us first derive a few general results. An important question is the maximum value of C, for given $\bar{\rho}$ and $\bar{\eta}$, allowing for an arbitrary variation of r and ϕ.

Varying r we find that C takes its maximum for

$$r = R_b \equiv \sqrt{\bar{\rho}^2 + \bar{\eta}^2}, \tag{52}$$

independently of ϕ. The resulting maximum $C_{\max}(\phi)$ at $r = R_b$ can be written as

$$C_{\max}(\phi) = \frac{\sin\gamma \sin\phi}{1 + \cos\gamma \cos\phi}, \tag{53}$$

and only depends on ϕ and γ. Viewed as a function of ϕ it can reach its absolute maximum $C = 1$ for $\cos\phi = -\cos\gamma$.

A useful representation is obtained by plotting contours of constant C in the $(r,\ \phi)$ plane, for given values of $\bar{\rho}$ and $\bar{\eta}$. This is illustrated in Fig. 6 for the standard model best-fit result $\bar{\rho} = 0.20$, $\bar{\eta} = 0.35$ [28]. Within the standard model this illustrates the correlations between the hadronic penguin parameters r and ϕ and direct CP-violation in $B \to \pi^+\pi^-$ decays. Upper limits on r and ϕ imply an

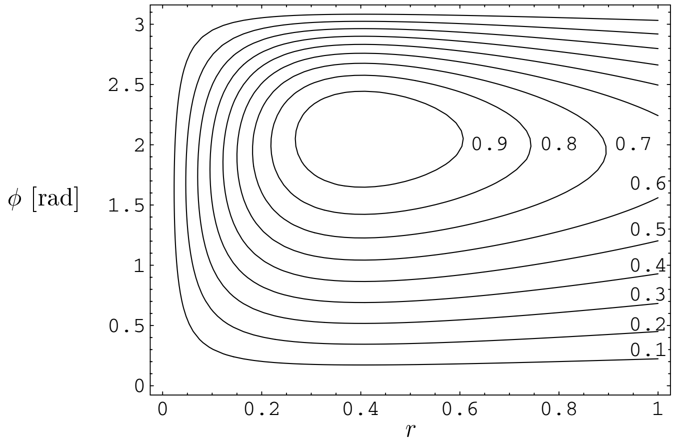

Fig. 6. Contours of constant C in the $(r,\ \phi)$ plane for fixed $\bar{\rho} = 0.20$ and $\bar{\eta} = 0.35$

upper limit on C unless they acquire unreasonably large values. For example, $r < 0.15$ and $\phi < 0.5$ yield $C < 0.21$.

We may relax the assumption of the validity of the standard model and discuss the parameter C from a different perspective. We consider the rather general scenario where new physics renders the standard unitarity triangle fit to determine γ invalid, while the extraction of R_b and the $B \to \pi^+\pi^-$ amplitudes remain essentially unaffected. In this situation it is convenient to slightly rewrite (6) as

$$C = \frac{2\kappa \sin\gamma \sin\phi}{1 + \kappa^2 + 2\kappa \cos\gamma \cos\phi}, \tag{54}$$

where we have introduced $\kappa \equiv r/R_b = |P/T|$. If we treat γ as unconstrained, we can still place an upper bound on C by maximizing C with respect to γ. Denoting this maximum by $\bar{C}$ we find

$$\bar{C} = \frac{2\kappa \sin\phi}{\sqrt{(1+\kappa^2)^2 - 4\kappa^2 \cos^2\phi}}, \tag{55}$$

where the maximum occurs at $\cos\gamma = -2\kappa\cos\phi/(1+\kappa^2)$.

If $\kappa = 1$, or equivalently $r = R_b$, then $\bar{C} \equiv 1$ independent of ϕ, and no useful upper bound is obtained. On the other hand, if $\kappa < 1$, then $\bar{C}$ is maximized for $\phi = \pi/2$. Under the general assumptions stated above and without any assumption on the strong phase ϕ we thus find the general bound

$$C < \frac{2\kappa}{1+\kappa^2}. \tag{56}$$

For the conservative bound $r < 0.15$, $\kappa < 0.38$ this implies $C < 0.66$. The bound on C can be strengthened by using information on ϕ, as well as on κ, and employing (55). Then $\kappa < 0.38$ and $\phi < 0.5$ gives $C < 0.39$.

6 Tests of factorization predictions

The analyses described above require theoretical input on the penguin parameter $r\exp(\mathrm{i}\phi)$. We have relied on model-independent calculations based on the heavy-quark limit of

QCD, including model estimates of subleading effects. To reinforce the validity of the approximations, in particular the large-m_b limit for realistic values of the b-quark mass, it is important to test other predictions, obtained within the same framework, against experiment. For this purpose it is necessary to keep in mind that both hadronic effects as well as effects from new physics could in general be the origin of any discrepancy. Both effects need to be disentangled as far as possible. Especially useful tests of the QCD aspects in hadronic B decays are those that have no, or very little, dependence on weak phases and potential new physics contributions. We shall discuss several such tests, which pertain to the essential ingredients for the P/T ratio $r\exp(\mathrm{i}\phi)$, namely the tree amplitude, the penguin amplitude and annihilation effects in $B \to \pi\pi$ decays.

6.1 Tree amplitude

The first example is a factorization test for the rate of the tree-type decay $B^\pm \to \pi^\pm\pi^0$ by taking a suitable ratio with the semileptonic rate of $B \to \pi l\nu$. For similar decays such observables have already been discussed e.g. in [33]. More recently it has been proposed to employ the decay $B \to \pi l\nu$ for an estimate of the tree amplitude in $B \to \pi^+\pi^-$, assuming factorization [34]. This analysis is similar to the one suggested here, however our main emphasis is somewhat different. We consider the factorization test as a cross-check of dynamical calculations based on the heavy-quark limit [4], rather than a way to determine the tree amplitude. The reason for this distinction is the fact that the tree contribution in $B^+ \to \pi^+\pi^0$ is not exactly the same as the one required for $B \to \pi^+\pi^-$. A related discussion can also be found in [35].

The differential decay rate for $B_d \to \pi^- l^+\nu$ is given by

$$\frac{\mathrm{d}\Gamma(B_d \to \pi^- l^+\nu)}{\mathrm{d}s} = \frac{G_\mathrm{F}^2 m_B^5}{192\pi^3}|V_{ub}|^2\,\lambda_\pi^{3/2}(s)\,f_+^2(q^2), \tag{57}$$

with

$$\lambda_\pi(s) = 1 + r_\pi^2 + s^2 - 2s - 2r_\pi - 2r_\pi s,$$
$$r_\pi = \frac{m_\pi^2}{m_B^2}, \quad s = \frac{q^2}{m_B^2}. \tag{58}$$

Here q^2 is the invariant mass of the lepton pair and $f_+(q^2)$ is a $B \to \pi$ transition form factor. Equation (57) is valid for leptons $l = e$, μ where the mass is negligible. The pion mass effect is also very small, $r_\pi = 7 \times 10^{-4}$, and can likewise be neglected. In this case the branching fractions of $B^+ \to \pi^+\pi^0$ and $B_d \to \pi^- l^+\nu$ are related through

$$B(B^+ \to \pi^+\pi^0) = 3\pi^2 f_\pi^2 |V_{ud}|^2 \left.\frac{\mathrm{d}B(B_d \to \pi^- l^+\nu)}{\mathrm{d}q^2}\right|_{q^2=0} \frac{\tau(B^+)}{\tau(B_d)} \times |a_1 + a_2|^2, \tag{59}$$

where a_1, a_2 are QCD coefficients [2,4]. In (59) the dependence on $|V_{ub}|$ and $f_+(0)$, which are not known precisely, have cancelled out. Once the branching fractions are measured, $|a_1 + a_2|$ can be extracted experimentally via (59) and compared with the theoretical prediction. This test is useful since neither QCD penguins nor weak annihilation corrections affect the $B^+ \to \pi^+\pi^0$ amplitude. It can thus provide us with a check on the tree amplitude $a_1 + a_2$, which includes non-trivial hard spectator interactions in QCD factorization, but does not depend on the other complications. Moreover, this test is independent of CKM phases and very unlikely to be modified by non-standard physics. It probes, as desired, crucial aspects of the QCD dynamics in $B \to \pi\pi$ decays.

In order to determine the differential semileptonic branching ratio $\mathrm{d}B_\mathrm{SL}/\mathrm{d}q^2 \equiv \mathrm{d}B(B_d \to \pi^- l^+\nu)/\mathrm{d}q^2$ at $q^2 = 0$, one needs to fit the q^2 spectrum of the semileptonic decay. We may use the expression

$$\frac{\mathrm{d}B_\mathrm{SL}}{\mathrm{d}q^2} = \left.\frac{\mathrm{d}B_\mathrm{SL}}{\mathrm{d}q^2}\right|_{q^2=0} \frac{\left(1 - \frac{q^2}{m_B^2}\right)^3 \left(1 + a\frac{q^2}{m_{B^*}^2}\right)^2}{\left(1 - \frac{q^2}{m_{B^*}^2}\right)^2}, \tag{60}$$

which follows from (57) and the parameterization of the form factor suggested in [34],

$$\frac{f_+(q^2)}{f_+(0)} = \frac{1 + a\frac{q^2}{m_{B^*}^2}}{1 - \frac{q^2}{m_{B^*}^2}}. \tag{61}$$

At present the data on $B_d \to \pi^- l^+\nu$ decays are not yet accurate enough to give a stringent test [34]. The situation should improve substantially in the future and will then yield valuable information on QCD dynamics in $B \to \pi\pi$ decays.

6.2 Penguin-to-tree ratio

The decay mode $B^+ \to \pi^+ K^0$ is essentially a pure penguin process, up to a negligible rescattering contribution [4]. The ratio of the $B^+ \to \pi^+ K^0$ to the $B^+ \to \pi^+\pi^0$ branching fraction is therefore a useful probe of the pegnuin-to-tree ratio [4, 35]. In analogy to the relevant parameter r in $B \to \pi^+\pi^-$, one may define a quantity $\tilde{r}$, which can be expressed through observables:

$$\tilde{r} \equiv \left|\frac{(a_4^c + r_\chi a_6^c + r_A b_3)_{\pi K}}{a_1 + a_2}\right| = \left|\frac{V_{ub}}{V_{cb}}\right| \frac{f_\pi}{f_K}\sqrt{\frac{B(B^+ \to \pi^+ K^0)}{2\,B(B^+ \to \pi^+\pi^0)}} = 0.12 \pm 0.01 \pm 0.01. \tag{62}$$

Here CP-averaged branching fractions are understood. The quoted number is derived from current experimental results, where the first error comes from $|V_{ub}/V_{cb}| = 0.10 \pm 0.01$ [28], the second from the branching ratios (see Table 3).

The parameter $\tilde{r}$ differs from r in the numerator through a different annihilation correction (b_3 instead of $b_3 + 2b_4$

Table 3. Current world average values for $B \to \pi\pi$, $K\pi$ branching ratios (CP-averaged, in units of 10^{-6}) [27]

$B^0 \to \pi^+\pi^-$	$B^+ \to \pi^+\pi^0$	$B^0 \to \pi^0\pi^0$
5.0 ± 0.4	5.5 ± 0.6	1.45 ± 0.29
$B^0 \to K^+\pi^-$	$B^+ \to K^+\pi^0$	$B^+ \to K^0\pi^+$
18.9 ± 0.7	12.1 ± 0.8	24.1 ± 1.3
$B^0 \to K^+K^-$	$B^+ \to K^+\bar{K}^0$	$B^0 \to K^0\bar{K}^0$
$0.05^{+0.10}_{-0.09}$	1.2 ± 0.3	$0.96^{+0.25}_{-0.24}$

(10)) and through small $SU(3)$ breaking differences in the light-meson distribution amplitudes (πK instead of $\pi\pi$). In the denominator $\tilde{r}$ has the pure tree amplitude $a_1 + a_2$, while r has a_1 corrected by small penguin and annihilation terms. Despite these differences in details, the structure of r and $\tilde{r}$ are very similar. In fact, the theoretical value for $\tilde{r}$ from QCD factorization,

$$\tilde{r} = 0.081 \pm 0.016 \pm 0.016 = 0.081 \pm 0.023, \qquad (63)$$

is very close to the corresponding value for r, and both are compatible with the experimental number in (62).

A final comment concerns the branching ratio for $B \to \pi^0\pi^0$, which appears to be somewhat larger experimentally (Table 3) than expected in recent theoretical calculations [35], even though the error bar is still large. It should be stressed that $B \to \pi^0\pi^0$, being color-suppressed (amplitude involving a_2), is highly sensitive to the dynamics of hard spectator interactions, which so far are only known to lowest order in QCD and depend on poorly known input within the factorization framework. These uncertainties strongly affect $B \to \pi^0\pi^0$, but are considerably smaller in $B \to \pi^+\pi^0$ and $B \to \pi^+\pi^-$, as already pointed out in [2]. In [35] a scenario with an enhanced $B \to \pi^0\pi^0$ rate, without the need for very unusual hadronic input, was suggested. Such a scenario with large a_2 could be checked using the factorization test discussed in the preceding subsection. We emphasize, however, that the uncertainties in a_2 specific to $B \to \pi^0\pi^0$ have essentially no impact on the penguin-to-tree ratio r exp(iϕ), because the dominant hadronic physics is characteristically different. Even a relatively large value of $B(B \to \pi^0\pi^0)$ does therefore not invalidate the theoretical results for r exp(iϕ).

6.3 Annihilation decays

Amplitudes from weak annihilation represent power-suppressed corrections, which are uncalculable in QCD factorization and so far need to be estimated relying on models [4]. At present there are no indications that annihilation terms would be anomalously large, but they do contribute to the theoretical uncertainty. Effectively, annihilation corrections may be considered as part of the penguin amplitudes. To some extent, therefore, they are tested with the help of the quantity $\tilde{r}$ discussed in the previous subsection. Nevertheless, in order to disentangle their impact from other effects it is of great interest to test annihilation separately. This can be done with decay modes that proceed through annihilation or at least have a dominant annihilation component.

Examples are the $B \to KK$ modes in Table 3. These, however, are CKM suppressed and have not been measured accurately at present. The $K^+\bar{K}^0$ and $K^0\bar{K}^0$ channels have both annihilation and penguin contributions. On the other hand $B \to K^+K^-$ is a pure weak annihilation process and therefore especially important. Further discussions can be found in [4, 35].

7 Conclusions

In this paper we have proposed strategies to extract information on weak phases from CP-violation observables in $B \to \pi^+\pi^-$ decays even in the presence of hadronic contributions related to penguin amplitudes. Our main results can be summarized as follows.

(1) An efficient use of mixing-induced CP-violation in $B \to \pi^+\pi^-$ decays, measured by S, can be made by combining it with the corresponding observable from $B \to \psi K_S$, $\sin 2\beta$ or $\tau = \cot\beta$.

(a) The unitarity triangle parameters $\bar{\rho}$ and $\bar{\eta}$ can then be obtained in closed form as functions of the observables τ, S and the hadronic penguin parameters r, ϕ (see (33) and (34)).

(b) The sensitivity on the hadronic quantities, which have typical values $r \approx 0.1$, $\phi \approx 0.2$, is very weak. In particular, there are no first-order corrections in ϕ. For moderate values of ϕ its effect is negligible.

(c) Neglecting ϕ, the dependence of $\bar{\eta}$ on r comes merely through an overall factor $(1+r)$. The impact of the uncertainty in $r \approx 0.1$ becomes clearly visible and is seen to be greatly reduced. A simple determination of the unitarity triangle from τ and S is thus possible (see (35)).

(2) The parameters $\bar{\eta}$, $1-\bar{\rho}$, R_t and γ are bounded from below, depending only on τ and S and essentially without relying on hadronic input (see (45)).

(3) The parameter of direct CP-violation C depends much stronger on hadronic input than S, but yields complementary information and can constrain r and ϕ within the standard model.

As an input to the phenomenological discussion we also studied the calculation of the penguin parameters r and ϕ in QCD.

(1) We have analyzed r and ϕ within QCD factorization with a particular view on theoretical uncertainties.

(2) $B \to \pi^+\pi^-$ amplitudes can be expanded simultaneously in $1/m_b$ and $1/N$, which leads to an interesting pattern of simplifications. All power corrections suffering from infrared end-point divergences in the factorization formalism are at least of second order in this double expansion, while the most important effects survive at linear order in $1/m_b$ or $1/N$.

(3) The different contributions to $B \to \pi^+\pi^-$ amplitudes, the tree component, the penguin-to-tree ratio and annihilation effects, appear in similar form in other B-decay channels, such as $B^+ \to \pi^+\pi^0$, $B^+ \to \pi^+K^0$ and $B \to K^+K^-$. These can be used to validate theoretical predictions, separately for the various components.

The results presented in this paper should be useful for interpreting the forthcoming experimental measurements of CP-violation in $B \to \pi^+\pi^-$ decays in a transparent way and help to achieve a reliable control over the theoretical uncertainties.

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